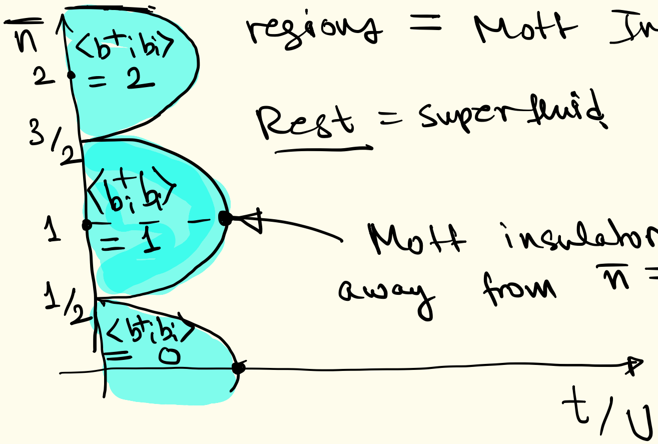


Blue shaded regions = Mott Insulator.

Rest = superfluid

Mott insulator less stable away from $\bar{n}$ = integer

Quantum phase transition when $\bar{n} \neq$ integer.

The easiest way to proceed is to work with coherent-state path integral for bosons in the grand canonical ensemble:

$$Z = \int db\, db^* \, \exp(-S)$$

$$S = \int_0^\beta d\tau \Big[\sum_i b^*_i \frac{\partial b_i}{\partial \tau} - \mu \sum_i b^*_i b_i + U \sum_i (b^*_i b_i)^2 - t \sum_{\langle ij \rangle} (b^*_i b_j + c.c.) \Big]$$

where b = complex number.

The order-parameter for the transition is $\langle b^{+} \rangle \equiv \psi$. Therefore, it makes sense to write the path-integral in terms of ψ.

One way to do this is to use a H-S (Hubburd - Stratanovich) field ~~to~~ to write.

$$S = \int_0^\beta d\tau \sum_i \left[b^* \frac{\partial b}{\partial \tau} - \mu b^*_i b_i + U (b^*_i b_i)^2 - \psi_i b^*_i - \psi^* b_i \right] + \sum_{\langle ij \rangle} \psi^*_i (t^{-1})_{ij} \psi_j$$

This action has the following 'gauge' invariance

$$b_i \rightarrow b_i e^{i\phi(\tau)}$$
$$\psi_i \rightarrow \psi_i e^{i\phi(\tau)}$$
$$\mu \rightarrow \mu + i \frac{\partial \phi}{\partial \tau}.$$

Next, one integrates out 'b' to obtain action solely in terms of ψ. This can be

done order-by-order perturbatively in U.
One obtains, based on symmetry grounds,

$$S = \int d^dx \int d\tau \Big[\underbrace{a \, \psi^* \frac{\partial \psi}{\partial \tau}}_{\text{Breaks Lorentz invariance.}} + \left| \frac{\partial \psi}{\partial \tau} \right|^2$$

$$+ |\nabla \psi|^2 + t\, |\psi|^2 + u\, |\psi|^4 \Big]$$

The coefficient 'a' may be fixed by requiring that S is invariant under the above guage transformation.

$$a\, \psi^* \frac{\partial \psi}{\partial \tau} \rightarrow a\, \psi^* e^{-i\varphi} \Big[\psi \, i \frac{\partial \varphi}{\partial \tau} e^{i\varphi} + \frac{\partial \psi}{\partial \tau} e^{i\varphi} \Big]$$

$$= a \psi^* \frac{\partial \psi}{\partial \tau} + i\, |\psi|^2 \frac{\partial \varphi}{\partial \tau} a$$

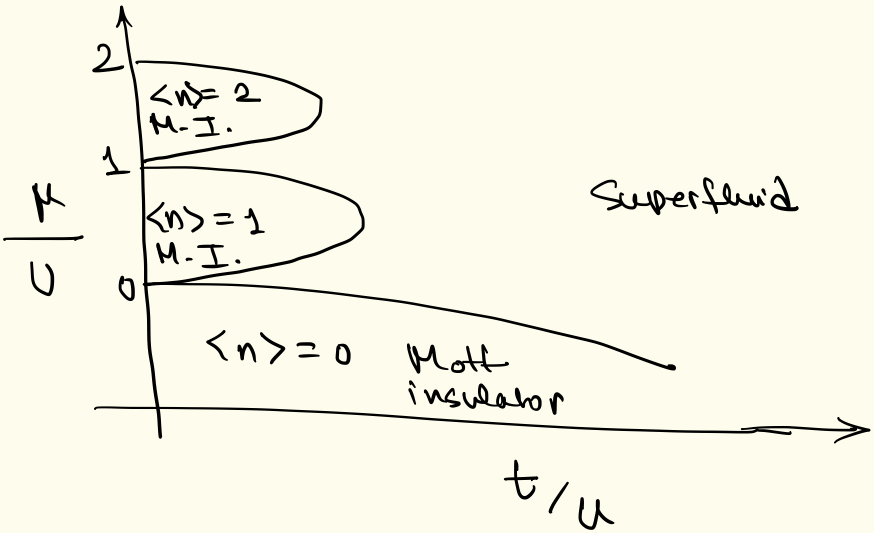
Therefore t must depend on μ in such a way that it cancels out this term.

$$t\, |\psi|^2 \rightarrow \Big[t + \frac{\partial t}{\partial \mu} \delta\mu \Big] |\psi|^2$$

$$= t\, |\psi|^2 + \frac{\partial t}{\partial \mu} \, i \frac{\partial \varphi}{\partial t} |\psi|^2.$$

$$\Rightarrow \quad a + \frac{\partial t}{\partial \mu} = 0 \quad \Rightarrow \quad a = -\frac{\partial t}{\partial \mu}$$

Plotting the phase diagram in the t, μ plane gives a very similar plot as the one in t, n plane.



The condition $\frac{\partial t}{\partial \mu} = 0$ is precisely the points which correspond to integer filling in the canonical ensemble.

$\Rightarrow \quad a = 0$ when $\bar{n} \in \mathbb{Z}$, consistent with our earlier analysis where we found that when $\bar{n} \in \mathbb{Z}$, the field theory has relativistic invariance in $x-\tau$ plane.

When $\bar{n} \notin \mathbb{Z}$ (i.e. $\mu \neq \mathbb{Z} + \frac{1}{2}$), the field theory has a term $\psi^* \frac{\partial \psi}{\partial \tau}$ and therefore space and time scale differently.

Lets do an RG analysis taking into account the anisotropic scaling in space time.

$$S = \int d^d x \, d\tau \, \left[\psi^* \partial_\tau \psi + |\nabla \psi|^2 + t|\psi|^2 + u|\psi|^4 \right]$$

where we have dropped the $|\partial_\tau \psi|^2$ term since it is subleading compared to $\psi^* \partial_\tau \psi$ and will turn out to be irrelevant.

We will do RG analysis by demanding that the kinetic-energy terms be invariant under rescaling of time and space.

Since the action is not relativistic invariant, one may suspect that the rescaling $x' = \frac{x}{b}$, $\tau' = \frac{\tau}{b}$ will not work. Indeed, the only way to make sure that

$$\psi^* \partial_{\tau'} \psi + |\vec{\nabla}_{x'} \psi|^2$$

$$\sim b^{\#} (\psi^* \partial_\tau \psi + |\vec{\nabla}_x \psi|^2)$$

is to use the scaling $x' = \frac{x}{b}$, $\tau' = \frac{\tau}{b^2}$

Under this transformation, the kinetic energy scales as

$$\int d^d x \, d\tau \, (\psi^* \partial_\tau \psi + |\vec{\nabla}_x \psi|^2)$$

$$= b^{d+2} \int d^d x' \, d\tau' \, \frac{[\psi^* \partial_{\tau'} \psi + |\vec{\nabla}_{x'} \psi|^2]}{b^2}$$

Therefore we choose $\psi(x, \tau) = \psi'(x, \tau') \, b^{-\frac{d}{2}}$ so that the kinetic energy is invarint. The scaling $x' = \frac{x}{b}$, $\tau' = \frac{\tau}{b^2}$ implies that the dynamical critical exponent is z. Here $z = 2$.

Let's check the relevance / irrelevance of the interactions as well the term we dropped above, namely, $\int d^d x \, d\tau \, |\partial_\tau \psi|^2$.

Interaction term is $u \int d^d x \, d\tau \, (|\psi|^2)^2$

$$= u \, b^{d+2} \int d^d x' \, d\tau' \, b^{-2d} (|\psi|^2)^2$$

$$\Rightarrow \quad u' = u \, b^{2-d}$$

$\Rightarrow$ interactions are irrelevant for $d > 2$ and marginal for $d = 2$.

Contrast this this ϕ^4- theory / $O(N)$ model in D total dimensions (i.e. $S = \int d^D x \left[(\nabla \phi)^2 + t\phi^2 + u\phi^4 \right]$) where interactions are irrelevant for $D > 4$.

The general statement is that when the dynamical critical exponent is z, the upper critical dimension is determined via

$$\underbrace{d}_{\text{space-dimension}} + \underbrace{z}_{\text{dynamical exponent}} = 4$$

space-dimension dynamical exponent.

Here $z=2 \Rightarrow d=2$ is the upper critical dimension. To see this criteria $d+z=4$ explicity, lets repete the above caludation with general z: $x' = \frac{x}{b}$, $\tau' = \frac{\tau}{b^z}$. We again demand that $\int d^d x \, d\tau \, |\vec{\nabla}_x \psi|^2$ is invariant.

$$\int d^d x \, d\tau \, |\vec{\nabla}_x \psi|^2 = b^{d+z-2} \int d^d x' \, d\tau' \, |\vec{\nabla}_{x'} \psi|^2$$

Thus, $\psi \, b^{\frac{d+z-2}{2}} = \psi'$ keeps this term invariant. $\Rightarrow \psi = \psi' \, b^{-\frac{(d+z-2)}{2}}$.

The interaction term is $u \int d^d x \, d\tau \, (|\psi|^2)^2$

$$= u \, b^{d+z} \, b^{-2(d+z-2)} \int d^d x' \, d\tau' \, (|\psi'|^2)^2$$

$$\Rightarrow u' = u \, b^{4-(d+z)} \Rightarrow \text{irrelevant for}$$

$$d+z > 4.$$

Lets next consider the term be dropped,

$K \int d^d x \, d\tau \, |\partial_\tau \psi|^2$

$$= K \, b^{d-z} \, b^{-(d+z-2)} \int d^d x' \, d\tau' \, |\partial_{\tau'} \psi'|^2$$

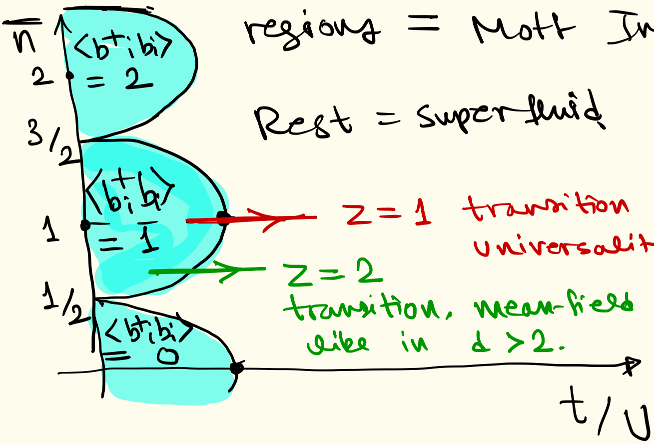
$$\Rightarrow K' = \frac{K}{b^{2z-2}} \Rightarrow \text{irrelevant for } z>1$$

as expected.

Summary

Blue shaded regions = Mott Insulator.

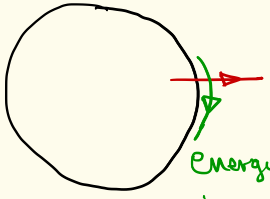
Rest = superfluid



$z = 1$ transition in $O(2)$ universality in $D = d+1$ dim.

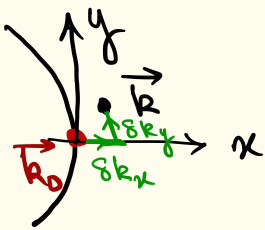
$z = 2$ transition, mean-field like in $d > 2$.

Introduction to RG for interacting fermions



Compared to the φ^4 like theory, where the low-energy modes live at a point, namely, $\vec{k}; \vec{\omega} \simeq \vec{0}$, in a system with a Fermi-surface, the low energy modes live on a surface (the Fermi surface that is). This makes the RG analysis much more challenging.

To understand the problem better, lets sit a point on the Fermi surface, $\vec{k} = \vec{k_0} = k_0 \hat{x}$ consider the energy of excitations nearby.



Excitation energy of $\vec{k} = \vec{k_0} + \delta \vec{k_x} + \delta \vec{k_y}$

$$= \frac{k^2}{2m} - \frac{k_0^2}{2m}$$

$$= \frac{\left[\left[(k_0 + \delta k_x) \hat{x} + \delta k_y \hat{y} \right]^2 - k_0^2 \right]}{2m}$$

Keeping terms to lowest non-zero order in $\delta k_x, \delta k_y$

$$\text{excitation energy} = \underbrace{\frac{k_0}{m}}_{= v_F \text{ (fermi velocity)}} \delta k_x + \frac{(\delta k_y)^2}{2m}$$

Lets define the operator that creates this excitation as $\psi^+(\vec{k}) \equiv c^+(\vec{k_0} + \vec{k})$ where c^+ is the standard fermion creation operator, and we have replaced δk by k for simplicity of notation.

Thus, the low-energy Hamiltonian close to $\vec{k_0}$ is

$$H = \psi^+(\vec{k}) \left[v_F \, k_x + \frac{k_y^2}{2m} \right] \psi(k)$$

In the path integral language, $\psi^+(k)$ will correspond to a Grassmanian field $\psi^*(k)$ and the low energy action, close to $\vec{k} = \vec{k_0}$ will be

$$S = \int d\tau \, dk_x \, d^{d-1} k_y \, \psi^* \left[\partial_\tau + v_F \, k_x + \frac{k_y^2}{2m} \right] \psi$$

$$= \int d\tau \, dx \, d^{d-1} y \, \psi^* \left[\partial_\tau - i v_F \, \partial_x - \frac{\partial_y^2}{2m} \right] \psi$$

Crucially, note that the k_y-direction is $d-1$ dimensional = dimension of the fermi surface.

The action knows the location of the fermi surface : $\omega = v_F k_x + \frac{k_y^2}{2m} = 0$

which defines a $d-1$ dimensional surface.

Now we are ready to define a RG scheme for this action. We restrict

$$v_F^2 k_x^2 + \left(\frac{k_y^2}{2m}\right)^2 < \Lambda^4$$ and rescale

all variables towards the single point $\vec{k}_0$ instead of rescaling towards all points on the fermi surface, which can be done but is a different RG procedure, see, R. Shankar, RG approach to interacting fermions. Since k_x and k_y appear with different powers in the action, we scale them differently, analogous to our discussion for $z=2$ bosons above:

$$y' = \frac{y}{b}, \quad x' = \frac{x}{b^2}, \quad \tau' = \frac{\tau}{b^2}.$$

So that

$$S = b^{2+d-1} \int d\tau' \, dx' \, d^{d-1}y' \; \psi^* [\partial_{\tau'} - i v_F \partial_{x'} - \frac{\partial^2_{y'}}{2m}] \psi$$

Thus, $\psi' = \psi \, b^{(d+1)/2}$ keeps the kinetic energy invariant.

Now we can inquire how do interactions flow under RG. The simplest interaction term involves four fermion operators, similar to the φ^4 theory for bosons.

$$u \int d\tau \, dx \, d^{d-1}y \; \psi^+ \psi^+ \psi \psi$$

Note that $(\psi^+)^2 = 0$ for fermions, but we also have a spin degree of freedom $\sigma = \uparrow / \downarrow$, Therefore, the four-fermion term is

$$u \int d\tau \, dx \, d^{d-1}y \; \psi^+_a \psi^+_b \psi_b \psi_a$$

where $a, b = \uparrow / \downarrow$.

Using the above RG scheme, this becomes,

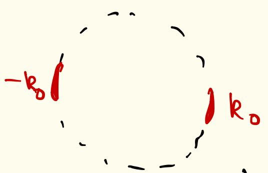
$$u \, b^{d-1+4} \, b^{-2(d+1)}$$

$$\int d\tau' \, dx' \, d^{d-1}y' \; \psi'^{+}_{a} \, \psi'^{+}_{b} \, \psi'_{b} \, \psi'_{a}$$

$$\Rightarrow \quad u' = u \, b^{-d+1}$$

$\Rightarrow$ interactions are irrelevant for $d > 1$ within this RG scheme.

This conclusion is indeed correct for weak interactions with one caveat. We are expanding around only one point on the Fermi surface. If one instead works with two such patches that are on the opposite sides of the Fermi surface



, then one finds that the interaction of kind $\psi^{+}_{\uparrow}(k_0)\,\psi^{+}_{\downarrow}(-k_0)\,\psi_{\uparrow}(k_0)\,\psi_{\downarrow}(-k_0)$ is marginally relevant. This corresponds to the superconducting instability.